# Enhancement of Noise-induced Escape through the Existence of a Chaotic Saddle

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(Dated: February 8, 2008)

We study the noise-induced escape process in a prototype dissipative nonequilibrium system, the Ikeda map. In the presence of a chaotic saddle embedded in the basin of attraction of the metastable state, we find the novel phenomenon of a strong enhancement of noise-induced escape. This result is established by employing the theory of quasipotentials. Our finding is of general validity and should be experimentally observable.

PACS number 05.45+b

arXiv:nlin/0309016v1 [nlin.CD] 4 Sep 2003

Since the seminal treatment of the noise-induced escape problem by Kramers [1], major progress has been made by Onsager and Machlup. They realized that the escape process consists of large fluctuations, which are very rare, and that the trajectory peaks sharply around some optimal (most probable) escape path [2]. Thus, despite the stochastic nature of the escape process, the escape path is of almost deterministic nature, as other paths than the most probable one have an exponentially smaller probability. That theory was derived for a small noise level $\delta \to 0$. A review on noise-induced escape in equilibrium systems and the most important recent advancements is given in [3].

In the last decade it has been realized, that systems that are not in thermal equilibrium or are lacking the property of detailed balance, can give rise to a large variety of interesting phenomena in the noise-induced escape problem. Only recently experiments on this problem have been conducted, using Josephson junctions, electronic circuits, lasers, and an electron in a Penning trap [4]. Some of the most interesting novel theoretical findings include a pre-exponential factor of the Kramers rate [5], a symmetry breaking bifurcation of the optimal escape path [6] and a distribution of the escape paths originating from a cusp point singularity [7]. Furthermore, the very intriguing phenomenon of saddle-point avoidance has been discovered [8]. For a fluctuating barrier the effect of resonant activation has been theoretically predicted [9] and experimentally confirmed [10]. Also a stepwise growth of the escape rate for short time scales has been found [11]. Recently, an oscillation of the escape rate in dependence on the friction for a multiwell potential was demonstrated [12]. For periodically driven systems a number of interesting results has been obtained as well, like a resonantly decrease in the activation energy [13], a logarithmic susceptibility of the fluctuation probability [14], time oscillations of escape rates [15] and enhancement of escape due to transient chaos [16].

Here we report on a new mechanism of lowering the required energy for noise-induced escape (enhancement of escape), thus a reduction of the mean first passage time. This happens, if a chaotic saddle is embedded in the open neighborhood of the basin of a metastable state. Then the escape trajectory does not only pass through a single unstable periodic orbit [17]. By contrast, it can pass through the chaotic saddle, i. e. a geometrically strange, invariant, non-attracting set (which is made up of an infinite number of unstable periodic orbits). The trajectory can jump between points of the chaotic saddle with no additional activation energy required. The overall lowering of the activation threshold is due to the fact, that the escape process consists now of three subsequent steps: Firstly, the trajectory jumps on one orbit on the chaotic saddle. Secondly, it switches on the chaotic saddle, without need of input energy, to select the orbit which allows the easiest escape, and thirdly it fluctuates from that orbit to the saddle point on the basin boundary. By this mechanism, the chaotic saddle is transformed into a dynamically relevant quantity, whereas in noisefree systems it is only important for transient behavior. In this way the chaotic saddle acts as a ‘shortcut’.

Since noise-induced escape has previously been studied using dissipative maps [18], which allow analysis in a straitforward way, we demonstrate our findings for the Ikeda map [19]. This is an idealized model of a laser pulse i n an optical cavity. With complex variables it has the form

$$z_{n+1} = a + b z_n \exp\left[i\kappa - \frac{i\eta}{1+|z_n|^2}\right], \qquad (1)$$

where $z_n = x_n + iy_n$ is related to the amplitude and phase of the $nth$ laser pulse exiting the cavity. The parameter $a$ is the laser input amplitude and corresponds to the forcing of the system. The damping $(1-b)$ accounts for the reflection properties of mirrors in the cavity and measures the dissipation. The empty cavity detuning is given by $\kappa$ and the detuning due to a nonlinear dielectric medium by $\eta$. The Ikeda map gives rise to rich dynamical behavior, exhibiting for some parameters even highly multistable behavior [20].

We fix the parameters at $a = 0.85, b = 0.9$ and $\kappa = 0.4$ and vary only $\eta$ in the range $2.6 < \eta < 12$. For the noiseless system two stable states are present. One fixed point (state A, $\Diamond$ in Fig. 1) undergoes a period doubling sce-

nario and becomes a chaotic attractor at $\eta \approx 5.5$ Another fixed point (state B, $\square$ in Fig. 1) remains a fixed point over the whole parameter range considered. The noise-induced escape from state A is investigated. The basin boundary separating these two stable states is a smooth curve which is build by the stable manifold of the saddle point C ($\ast$ in Fig. 1), separating the two stable states. With the above mentioned features of the system, there

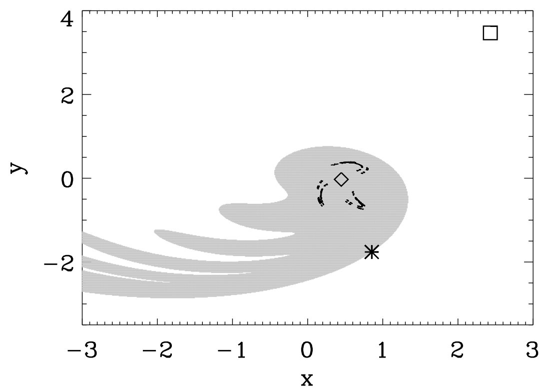


FIG. 1: Grey dots represent the basin of attraction for the fixed point marked with a $\Diamond$ for $\eta = 4.1$. The other fixed point is also depicted ($\square$). The chaotic saddle is shown with black dots and the saddle point on the basin boundary is marked by $\ast$.

are no unusual effects expected in the noise-induced escape problem. However, for a critical value of $\eta_c = 3.5686$ an additional period 3 solution close to the fixed point (state A) emerges, with a fractal basin boundary between these two solutions and a chaotic saddle embedded in this fractal basin boundary. It is important to note, that the basin boundary between the two fixed points remains smooth over the whole parameter range considered here. Increasing $\eta$ further, the stable period 3 solution dissapears in a boundary crisis, yet a chaotic saddle is *still* present beyond the boundary crisis, completely embedded in the open neighborhood of the basin of the stable solution, as can be seen in Fig. 1 for $\eta = 4.1$. A chaotic saddle is a geometrically strange, invariant, non-attracting set. It is computed using the PIM-triple algorithm [21]. We stress that it is the chaotic saddle, that has a remarkable effect on the average escape time from the stable fixed point.

To treat the problem of noise-induced escape we now employ the theory of quasipotentials, which gives rigorous results on the influence of noise on the invariant density and the mean first passage time. Quasipotentials have been introduced in the mathematical literature for time-continuous systems in [22] and for discrete time ones in [23]. For systems of physical interest, they were first proposed in [25] and extended to systems with coexisting attractors in [26]. Discrete systems with strange invariant sets were for the first time treated in [27]. Quasipotentials can be derived through a minimization procedure of the action of escape trajectories from a Hamilton-Jacobi equation [24]. The action to be minimized has the form

$$S_N[(z_i)_{0\leq i<N}] = \frac{1}{2}\sum_{i=0}^{N-1}[z_{i+1} - f(z_i)]^2, \qquad (2)$$

for the map $z_{n+1} = f(z_n) + \sigma\,\xi_n$, where $\sigma$ is the standard deviation of the additive, Gaussian, white noise term $\xi_n$. With appropriate boundary conditions the infimum of this action with respect to $N$ and $i$ along a path is the quasipotential $\Phi$. The mean first exit time is then given in analogy to Kramer's law:

$$\langle\tau\rangle \sim \exp\left[\frac{\Delta\Phi}{\sigma^2}\right], \qquad (3)$$

whith $\Delta\Phi$ defined as the minimal quasipotential difference

$$\Delta\Phi := \inf\{\Phi(y) - \Phi(a) : a \in A, y \in \partial G\}, \qquad (4)$$

where $A$ is the attractor and $\partial G$ is the basin boundary. In Fig. 2 the quasipotential is shown for $\eta = 4.1$. There is a single peak corresponding to the fixed point A and a plateau region of a practically constant quasipotential, which reflects the chaotic saddle. Employing

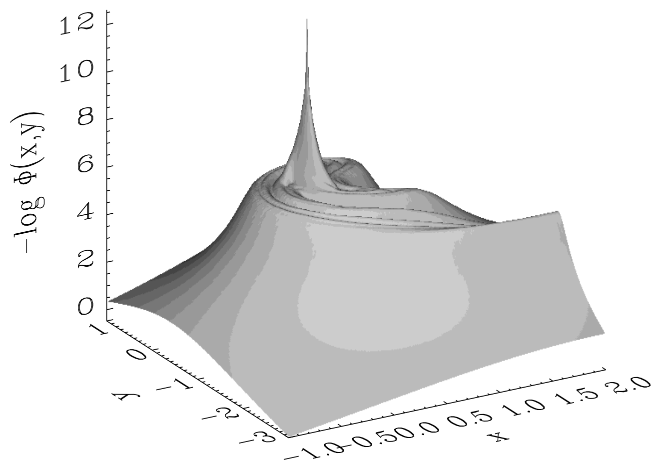


FIG. 2: Quasipotential $\Phi(x, y)$ for the Ikeda map with $\eta = 4.1$ on a $300 \times 300$ grid. The single peak corresponds to the fixed point. Also, an extended plateau at $-\log\Phi(x, y) \approx 5.0$ is visible, caused by the chaotic saddle.

the quasipotential for the noise-induced escape problem, the minimum value of $\Phi(x, y)$ on the basin boundary has to be determined. This is exactly the minimum escape energy $\Delta\Phi(x, y)$, since the quasipotential at the stable solution (fixed point, periodic orbit or chaotic attractor) is zero. The point on the basin boundary, where this happens, is generally a saddle point of the system.

To quantify the escape process with the quasipotential, we plot for various values of $\eta$ the corresponding minimal escape energy $\Delta\Phi(x, y)$ in Fig. 3. To elucidate the role of the chaotic saddle as the origin of an enhancement of noise-induced escape, we also include in the plot the value of the height of the plateau in the quasipotential.
In the framework of quasipotentials, the difference in height of the escape energy and the saddle plateau corresponds to the distance between the basin boundary and the chaotic saddle, whereas the height of the saddle is related to the distance between the attractor and the saddle.
The mechanism of the escape process is closely connected to the existence of an embedded chaotic saddle. It consists of two steps, namely, a noise-induced fluctuation from the attractor (state A) to the chaotic saddle, and then from the chaotic saddle to the fixed point (state B) via the saddle point on the boundary. The escape can also be incomplete, as the trajectory may fall back from the chaotic saddle to the attractor. In a successful escape, the chaotic saddle acts as a 'shortcut', as its presence lowers the overall escape energy. This behavior seems to be especially pronounced if the chaotic saddle is closer to the basin boundary than to the attractor (compare the region $3.6 \leq \eta \leq 4.5$ of Fig. 3). Let us note that for $\eta = 5.5$ it is not clear, if there exists a chaotic saddle, which is the case for all other values of $\eta \geq 3.5686$ we have tested. The PIM-triple method, as well as the quasipotential, yield for $\eta = 5.5$ no conclusive result, as a chaotic saddle may exist *very close* to the chaotic attractor and numerically it is very difficult to distinguish between the two.

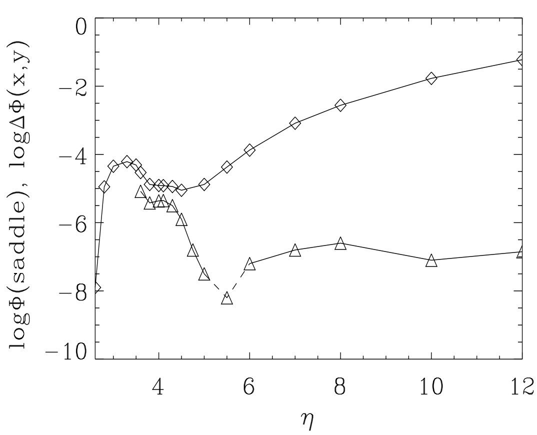


FIG. 3: Minimal escape energy ($\Diamond$) and height of the saddle plateau ($\triangle$) in logarithmic scale versus $\eta$.

It is important to quantify the influence of the relative sizes of the basins of attraction, since it is increasing with increasing $\eta$ and we are here only interested in the change of activation energy caused by the chaotic saddle. The distance between the attractor and the saddle point on the boundary is usually proportional to the relative size of the basin. Both quantities are expected to play a role in the stability of the metastable state located in the basin, although we are not aware of any theoretical work dealing with this relation directly. To compensate for the change of escape energy caused by the increase in size of the basin of attraction, in Fig. 4 the escape energy is divided by the ratio of the size of the basin of attraction of state A to the overall area (A + B) for 3 different sections of the phase space. The sections of the phase space decrease from top to bottom. The combination of the two quantities, potential height and basin size, yields a pronounced minimum at $\eta \approx 5.0$ for all 3 curves, thus confirming the essential role in lowering the escape energy played by the chaotic saddle. The most probable

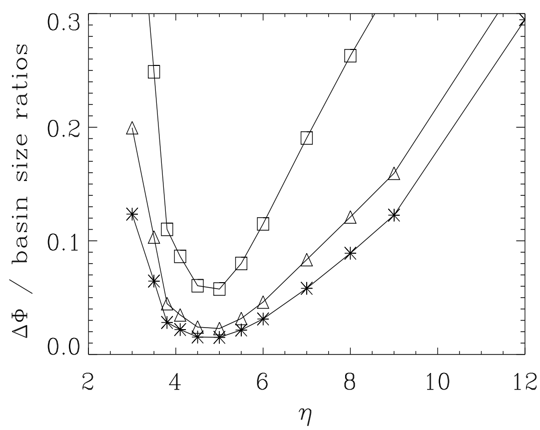


FIG. 4: Average quasipotential height (= escape energy) divided by the ratios of the sizes of the basins of attraction (# initial conditions A / (A + B)). The curves correspond to a smaller frame of reference from top to bottom, with the values $x \in [-10.0, 10.0], y \in [-10.0, 10.0]$ (marked with $\Box$), $x \in [-5.0, 5.0], y \in [-7.0, 7.0]$ (marked with $\triangle$), $x \in [-3.0, 2.0], y \in [-3.5, 4.0]$ (marked with $\ast$), respectively.

escape path [2, 22] for $\eta = 5.0$ is shown, together with the chaotic saddle, in Fig. 5. For this parameter value, there is a stable period 4 solution. As can be seen, the trajectory jumps at first directly on points of the chaotic saddle, moves secondly along points of the chaotic saddle for some iterations, until it is thirdly transported close to the basin boundary to the saddle point. Since the first step (from fixed point to saddle) and the last step (from saddle to the basin boundary) are minimal in this case, the enhancement is maximal. Other values of the Ikeda map, where no chaotic saddle is present, have been investigated as well. For these parameter values the effect could not be found, and the graphs corresponding to Fig. 4 have a strictly monotonic shape. This demonstrates that the existence of the chaotic saddle is of crucial importance for the occurrence of the enhancement of noise-induced escape.
Moreover, the stability of a fixed point is determined by

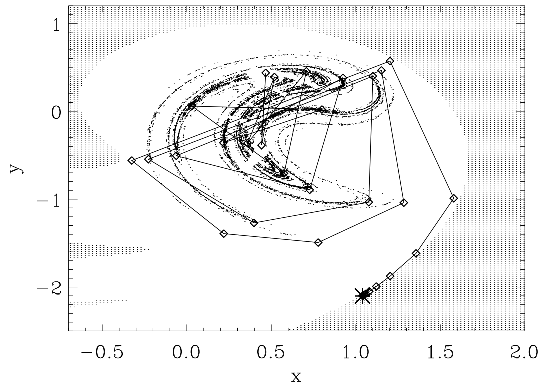


FIG. 5: Escape path and chaotic saddle for $\eta = 5.0$. The saddle point on the basin boundary is shown as $*$.

its eigenvalues. The eigenvalues are found to be $\lambda = 0.9$ for the whole range, where it exists. Consequently, the linear approximation is of no relevance to the noise-induced escape problem, as its range of validity is much smaller than the region for the escape, which is the whole open set of the basin of attraction shown in Fig. 1.

To conclude, we have demonstrated the effect of enhancement of noise-induced escape through the existence of a chaotic saddle in the open neighborhood of the metastable state for the Ikeda map as a parameter is varied. Employing the theory of quasipotentials, it was possible to understand this lowering of the escape threshold. We stress that the reported mechanism of the lowering of the escape energy is of qualitatively different nature from a recently found effect, where also an enhancement of noise-induced escape through transient motion (typical for chaotic saddles) has been found [16]. In this scenario, a nonadiabatically, periodically driven system exhibits a facilitation of noise-induced interwell transitions. This occurs, because the basin boundary becomes fractal and the distance between the two states is effectively reduced. In the mechanism reported here we always have a smooth basin boundary between the two states and the chaotic saddle is embedded in the basin of one state, not in the basin boundary between the states. The analysis of the exact escape path on the chaotic saddle, in contrast to the case where the trajectory leaves via a single periodic orbit [17] will be the presented in a much broader hashion in a future publication [28]. The reported new phenomenon is of general relevance for many physical and chemical problems. It is predicted to occur in a variety of systems, and should experimentally be observable.

We acknowledge A. Hamm and D. Luchinsky for valuable discussions and A. Hamm also for the help in programming. This work was supported by the DFG and INTAS.

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